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International Baccalaureate®
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FURTHER MATHEMATICS
STANDARD LEVEL
PAPER 2

Friday 21 May 2010 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

6 pages

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. All students should therefore be advised to show their working.

1. *[Maximum mark: 32]*

The binary operator $*$ is defined for $a, b \in \mathbb{R}$ by $a * b = a + b - ab$.

(a) (i) Show that $*$ is associative.

(ii) Find the identity element.

(iii) Find the inverse of $a \in \mathbb{R}$, showing that the inverse exists for all values of a except one value which should be identified.

(iv) Solve the equation $x * x = 1$. *[15 marks]*

(b) The domain of $*$ is now reduced to $S = \{0, 2, 3, 4, 5, 6\}$ and the arithmetic is carried out modulo 7.

(i) Copy and complete the following Cayley table for $\{S, *\}$.

$*$	**0**	**2**	**3**	**4**	**5**	**6**
0	0	2	3	4	5	6
2	2	0	6	5	4	3
3	3					
4	4					
5	5					
6	6					

(ii) Show that $\{S, *\}$ is a group.

(iii) Determine the order of each element in S and state, with a reason, whether or not $\{S, *\}$ is cyclic.

(iv) Determine all the proper subgroups of $\{S, *\}$ and explain how your results illustrate Lagrange's theorem.

(v) Solve the equation $2 * x * x = 5$. *[17 marks]*

2. *[Total mark: 16]*

Part A *[Maximum mark: 9]*

The points D, E, F lie on the sides [BC], [CA], [AB] of the triangle ABC and [AD], [BE], [CF] intersect at the point G. You are given that $CD = 2BD$ and $AG = 2GD$.

(a) By considering (BE) as a transversal to the triangle ACD, show that

$$\frac{CE}{EA} = \frac{3}{2}.$$

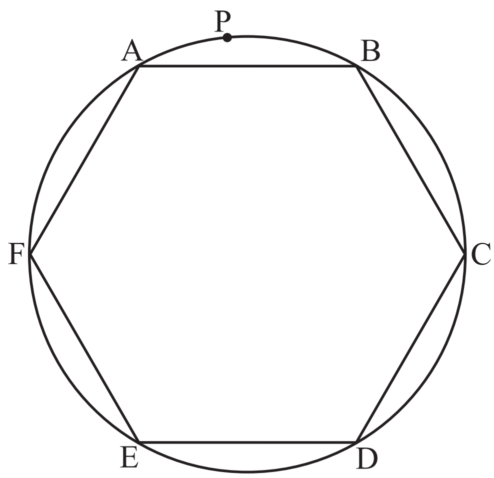
[2 marks]

(b) Determine the ratios

(i) $\frac{AF}{FB}$;

(ii) $\frac{BG}{GE}$. *[7 marks]*

Part B *[Maximum mark: 7]*

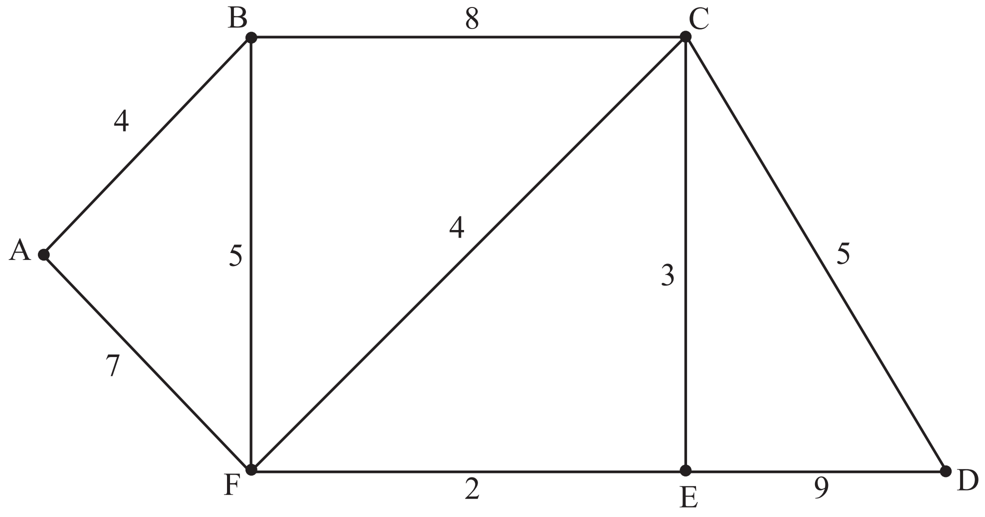


The diagram shows a hexagon ABCDEF inscribed in a circle. All the sides of the hexagon are equal in length. The point P lies on the minor arc AB of the circle. Using Ptolemy's theorem, show that

$$PE + PD = PA + PB + PC + PF.$$

3. *[Maximum mark: 18]*

The following diagram shows a weighted graph G.



(a) (i) Explain briefly what features of the graph enable you to state that G has an Eulerian trail but does not have an Eulerian circuit.

(ii) Write down an Eulerian trail in G. *[3 marks]*

(b) (i) Use Kruskal's algorithm to find and draw the minimum spanning tree for G. Your solution should indicate the order in which the edges are added.

(ii) State the weight of the minimum spanning tree. *[5 marks]*

(c) Use Dijkstra's algorithm to find the path of minimum total weight joining A to D, and state its weight. Your solution should indicate clearly the use of this algorithm. *[10 marks]*

4. *[Maximum mark: 13]*

(a) The weights, X grams, of tomatoes may be assumed to be normally distributed with mean μ grams and standard deviation σ grams. Barry weighs 21 tomatoes selected at random and calculates the following statistics.

$$\sum x = 1071; \sum x^2 = 54705$$

(i) Determine unbiased estimates of μ and σ^2.

(ii) Determine a 95 % confidence interval for μ. *[8 marks]*

(b) The random variable Y has variance σ^2, where $\sigma^2 > 0$. A random sample of n observations of Y is taken and S_{n-1}^2 denotes the unbiased estimator for σ^2. By considering the expression

$$\mathrm{Var}(S_{n-1}) = \mathrm{E}(S_{n-1}^2) - \{\mathrm{E}(S_{n-1})\}^2,$$

show that S_{n-1} is not an unbiased estimator for σ. *[5 marks]*

5. *[Maximum mark: 19]*

After a shop opens at 09:00 the number of customers arriving in any interval of duration t minutes follows a Poisson distribution with mean $\frac{t}{10}$.

(a) (i) Find the probability that exactly five customers arrive before 10:00.

(ii) Given that exactly five customers arrive before 10:00, find the probability that exactly two customers arrive before 09:30. *[7 marks]*

(b) Let the second customer arrive at T minutes after 09:00.

(i) Show that, for $t > 0$,

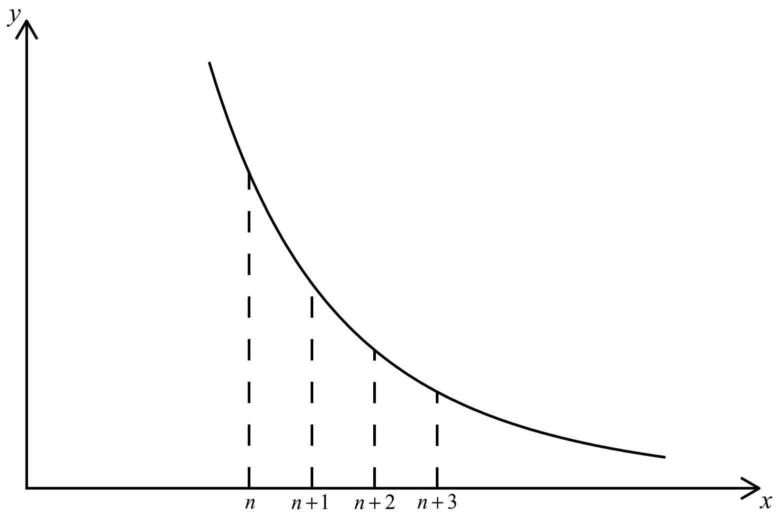
$$\mathrm{P}(T > t) = \left(1 + \frac{t}{10}\right)\mathrm{e}^{-\frac{t}{10}}.$$

(ii) Hence find in simplified form the probability density function of T.

(iii) Evaluate $\mathrm{E}(T)$.
(You may assume that, for $n \in \mathbb{Z}^+$ and $a > 0$, $\lim_{t \to \infty} t^n \mathrm{e}^{-at} = 0$.) *[12 marks]*

6. *[Maximum mark: 22]*

(a) The diagram shows a sketch of the graph of $y = x^{-4}$ for $x > 0$.



By considering this sketch, show that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=n+1}^{\infty} \frac{1}{r^4} < \int_n^{\infty} \frac{dx}{x^4} < \sum_{r=n}^{\infty} \frac{1}{r^4}.$$

[5 marks]

(b) Let $S = \sum_{r=1}^{\infty} \frac{1}{r^4}$.

Use the result in (a) to show that, for $n \geq 2$, the value of S lies between

$$\sum_{r=1}^{n-1} \frac{1}{r^4} + \frac{1}{3n^3} \text{ and } \sum_{r=1}^{n} \frac{1}{r^4} + \frac{1}{3n^3}.$$

[8 marks]

(c) (i) Show that, by taking $n = 8$, the value of S can be deduced correct to three decimal places and state this value.

(ii) The exact value of S is known to be $\frac{\pi^4}{N}$ where $N \in \mathbb{Z}^+$. Determine the value of N.

[6 marks]

(d) Now let $T = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^4}$.

Find the value of T correct to three decimal places. *[3 marks]*